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A VARIATIONAL METHOD FOR THE OPTIMIZATION OF

INTERPLANETARY ROUND-TRIP TRAJECTORIES

By John S. MacKay and Leonard G. Rossa

Lewis Research Center
Cleveland, Ohio

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SUMMARY

The indirect method of the calculus of variations is used to optimize inter-planetary round-trip trajectories for the case of a single, central, attracting body. The method of solution makes use of certain partial derivative properties of the Lagrangian multipliers associated with the Mayer formulation of the variational problem. This property of the multipliers allows the construction of mathematical expressions for certain other partial derivatives that must vanish when an optimum round trip has been found. These expressions are developed for the cases of propulsion systems using (1) fixed thrust and specific impulse or (2) variable thrust and constant exhaust jet power. Two numerical examples demonstrate how the analytical results may be applied to the solution of round-trip problems including (1) actual three-dimensional planetary positions and (2) planetocentric maneuvers.

INTRODUCTION

Failure to optimize the flight trajectory when planning interplanetary missions can result in substantial penalties in vehicle performance. For low-acceleration vehicles, such as those employing electric propulsion systems, it is desirable to vary the thrust vector optimally with time in order to achieve maximum performance. Solutions for optimum one-way journeys have been obtained by using both the direct and indirect methods of the calculus of variations. Several examples of solutions by the indirect method, with which this report is concerned, are given in references 1 to 5. In order to apply such solutions to the round-trip problem, the outbound trip must be combined with a similar inbound trip in such a way that, at return to Earth, some specified parameter (e.g., payload) is maximized. Systematic trial-and-error procedures for doing this can be found in references 4 and 5.

This report presents a variational solution for the complete round-trip problem. The method of solution makes use of certain variables needed for the solution of the inbound and outbound trips to identify the characteristics of the overall optimum for the round trip. Specifically, it is shown that it is possible to construct such partial derivatives as that of return mass with respect to the outbound travel time.

The method discussed herein is based on the fact that the Lagrangian multipliers, as used in the Mayer formulation, are the partial derivatives of the function to be extremized with respect to the problem variables. (An explanation of this characteristic of the multipliers can be found in ref. 6.) This fact may be applied to a variety of different problems. For example, the method is applied to a three-dimensional, two-body round-trip transfer using an electric propulsion system with either (1) fixed thrust and specific impulse or (2) variable thrust and constant jet power. Also, the problem of including the effects of planetocentric maneuvers is considered in both cases.

In order to demonstrate how the analytical results derived may be used in specific numerical problems, two Earth-Mars round-trip calculations have been made. The first of these omits the effects of planetocentric maneuvers but does illustrate the usefulness of the suggested criteria for the identification of an optimum round trip in the case in which three-dimensional ephemeris data is used. The second example considers a two-dimensional transfer between circular orbits with the planetocentric maneuvers included.

SYMBOLS

a thrust acceleration, m/sec^2

C first integral of Euler-Lagrange equations

c jet velocity, m/sec

D_i the operator, $\partial/\partial\lambda_i(0)$

E Weierstrass excess function

F $\sum\limits_{i=1}^{10} \lambda_i f_i$

f_i constraint relation

g function of initial and final conditions

J functional to be made an extremal

K constant of integration

m mass, kg

P power, w

P_j jet power, w

t time, sec

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t_w waiting time

u,v,w components of heliocentric velocity vector

V gravitational potential

x,y,z coordinates in Cartesian, inertial reference system, m

α powerplant specific weight, kg/w

β mass-flow rate, kg/sec

γ dummy variable

$d\gamma$ arbitrary differential in any variable γ

$\delta\gamma_i$ variation in γ_i at constant time

η propulsion-system power efficiency

θ angle between thrust vector and x,y-plane, radians

κ switching function

Λ $\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}$

λ Lagrangian multiplier

μ gravitational field strength, m^3/sec^2

ψ angle between thrust vector and x-axis, radians

Subscripts:

f   final

L   payload

max   maximum

p   planet

pp   powerplant

s   supply

0   initial

Superscripts:

.   d( )/dt

—   taken from planetocentric trajectory calculations

ANALYSIS

The method to be presented is best explained with the aid of an example; however, in order to create sufficient framework for a round-trip example, it is necessary to consider first a typical solution of the one-way trip by the indirect method of the calculus of variations.

Solution of the One-Way Trip

The problem to be solved here is that of finding the thrust direction and magnitude, as functions of time, which minimize the propellant consumed for a one-way interplanetary transfer satisfying specific initial and final conditions. Before this can be done, however, the propulsion system must be confined to some desired mode of operation. One mode commonly used is that of continuously variable thrust at constant jet power $P_j = \beta c^2/2$. Solutions using this constraint are given in references 1, 3, and 5; however, for the main development given here, it is assumed that the thrustor operates at a constant jet velocity and has two alternative choices for the propellant-flow rate, that is,

$$c = \text{const}$$

$$\beta = \beta_{max} \text{ or } 0$$

where c is the jet velocity and β the propellant-flow rate. Other developments using these constraints may be found in references 2 and 4. The variable-thrust mode is considered later as a modification of the main presentation.

Rather than minimize the propellant consumption, the problem is solved by minimizing the negative of the final mass, subject to the following constraints (see fig. 1):

$$f_1 = \dot{u} + V_x - \left(\frac{c\beta}{m}\right) \cos \psi \cos \theta = 0 \tag{1a}$$

$$f_2 = \dot{v} + V_y - \left(\frac{c\beta}{m}\right) \sin \psi \cos \theta = 0 \tag{1b}$$

$$f_3 = \dot{w} + V_z - \left(\frac{c\beta}{m}\right) \sin \theta = 0 \tag{1c}$$

$$f_4 = \dot{x} - u = 0 \tag{1d}$$

$$f_5 = \dot{y} - v = 0 \tag{1e}$$

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$$f_6 = \dot{z} - w = 0 \tag{1f}$$

$$f_7 = \dot{m} + \beta = 0 \tag{1g}$$

$$f_8 = \beta(\beta_{max} - \beta) = 0 \tag{1h}$$

$$f_9 = \dot{c} = 0 \tag{1i}$$

$$f_{10} = \dot{\beta}_{max} = 0 \tag{1j}$$

where $V(x,y,z) = -\mu/(x^2 + y^2 + z^2)^{1/2}$ is the gravitational potential and $V_x = \partial V/\partial x$, $V_y = \partial V/\partial y$, and so forth. The expressions f_1 to f_6 are the two-body equations of motion in three dimensions; the remaining expressions are related to the constraints imposed on the thrust device.

When this is formulated as a Mayer problem (ref. 7), the functional to be minimized is

$$J = g + \int_{t_0}^{t_f} \sum_{i=1}^{10} \lambda_i f_i \, dt = -m_f + \int_{t_0}^{t_f} F \, dt \tag{2}$$

where g (which is $-m_f$ for this problem) is some function of the initial and final conditions only. The Euler-Lagrange equations for this problem are

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{\gamma}_j}\right) = \frac{\partial F}{\partial \gamma_j} \qquad j = 1, 2, \ldots, 12 \tag{3}$$

where the γ_j are the problem variables u, v, w, x, y, z, m, β, c, β_{max}, ψ, and θ. More specifically, equations (3) are

$$\dot{\lambda}_1 = -\lambda_4 \tag{4a}$$

$$\dot{\lambda}_2 = -\lambda_5 \tag{4b}$$

$$\dot{\lambda}_3 = -\lambda_6 \tag{4c}$$

$$\dot{\lambda}_4 = \lambda_1 V_{xx} + \lambda_2 V_{xy} + \lambda_3 V_{xz} \tag{4d}$$

$$\dot{\lambda}_5 = \lambda_1 V_{yx} + \lambda_2 V_{yy} + \lambda_3 V_{yz} \tag{4e}$$

$$\dot{\lambda}_6 = \lambda_1 V_{zx} + \lambda_2 V_{zy} + \lambda_3 V_{zz} \tag{4f}$$

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$$\dot{\lambda}_7 = \frac{c\beta}{m^2} \left(\lambda_1 \cos \psi + \lambda_2 \sin \psi\right)\cos \theta + \lambda_3 \sin \theta \Big] \tag{4g}$$

$$\lambda_8(\beta_{max} - 2\beta) - \frac{c\left[(\lambda_1 \cos \psi + \lambda_2 \sin \psi)\cos \theta + \lambda_3 \sin \theta\right]}{m} + \lambda_7 = 0 \tag{4h}$$

$$\dot{\lambda}_9 = - \frac{\beta\left[(\lambda_1 \cos \psi + \lambda_2 \sin \psi)\cos \theta + \lambda_3 \sin \theta\right]}{m} \tag{4i}$$

$$\dot{\lambda}_{10} = \beta\lambda_8 \tag{4j}$$

$$\frac{c\beta}{m} (\lambda_2 \cos \psi - \lambda_1 \sin \psi)\cos \theta = 0 \tag{4k}$$

$$\frac{c\beta}{m} \left[\lambda_3 \cos \theta - (\lambda_1 \cos \psi + \lambda_2 \sin \psi)\sin \theta \right. = 0 \tag{4l}$$

If $\beta \neq 0$ and $\cos \theta \neq 0$, then equation (4k) gives

$$\tan \psi = \frac{\lambda_2}{\lambda_1}$$

and

$$\cos \psi = \frac{\lambda_1}{\pm \sqrt{\lambda_1^2 + \lambda_2^2}}$$

$$\sin \psi = \frac{\lambda_2}{\pm \sqrt{\lambda_1^2 + \lambda_2^2}}$$

Also, equation (4l) gives

$$\tan \theta = \frac{\lambda_3}{\pm \sqrt{\lambda_1^2 + \lambda_2^2}}$$

Thus, the three-direction cosines for the thrust acceleration are

$$\cos \psi \cos \theta = \frac{\lambda_1}{\pm \Lambda}, \quad \sin \psi \cos \theta = \frac{\lambda_2}{\pm \Lambda}, \quad \sin \theta = \frac{\lambda_3}{\pm \Lambda} \tag{5}$$

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where

$$\Lambda = \sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}$$

An application of the Weierstrass-Erdmann Corner Condition (ref. 7),

$$\left(\frac{\partial F}{\partial \dot{r}_j}\right)_+ = \left(\frac{\partial F}{\partial \dot{r}_j}\right)_- \qquad j = 1, 2, \ldots, 12 \tag{6}$$

indicates that all the Lagrangian multipliers except λ_8 are continuous across any corners that may occur in the trajectory. For this problem, corners will occur when either the sign of Λ or the value of β is changed.

The Weierstrass "E" test may be used to find the appropriate sign for Λ, as well as the value for β at each instant of time. This test states that, for a minimum of J,

$$0 \leq E = F^* - F - \sum_{j=1}^{12} \frac{\partial F}{\partial \dot{r}_j} \left(\dot{r}_j^* - \dot{r}_j\right) \tag{7}$$

where the starred functions differ from the unstarred functions by finite but admissible amounts. The unstarred functions, furthermore, are assumed to be the minimizing functions. The only quantities allowed such strong variations in this problem are Λ (which may change sign) and β. Thus, equation (7) can be written as:

$$0 \leq \beta\kappa - \beta^*\kappa^* \tag{8}$$

where

$$\kappa \quad \text{or} \quad \kappa^* = \pm \frac{c\Lambda}{m} - \lambda_7$$

For $\beta = \beta^*$ equation (8) reduces to

$$0 \leq \beta(\kappa - \kappa^*) = \frac{c\beta}{m} (\Lambda - \Lambda^*)$$

from which, for $\beta \neq 0$, it can be seen that the positive sign must be chosen for Λ. Also, if $\kappa = \kappa^*$, then equation (8) shows that $0 \leq (\beta - \beta^*)\kappa$; it follows, then, that $\beta = \beta_{max}$ when $\kappa > 0$ and $\beta = 0$ when $\kappa < 0$.

Finally, the transversality relation for this type of problem may be written as (ref. 6)

$$\left[\left(F - \sum_{j=1}^{12} \frac{\partial F}{\partial \dot{\gamma}_j} \dot{\gamma}_j\right) dt + \sum_{j=1}^{12} \frac{\partial F}{\partial \dot{\gamma}_j} d\gamma_j\right]_{t_0}^{t_f} - dm_f = 0 \qquad (9)$$

Depending on the type of boundary conditions imposed on the trajectory, this relation will give various additional boundary conditions that must be satisfied. For example, if a known value for one of the variables is desired at a boundary, then its differential in equation (9) is zero; otherwise, the coefficient of its differential must be zero and, thus, becomes an additional boundary condition.

For the kind of problem formulated here, where the function F does not depend explicitly on time, the coefficient of dt is a constant along the flight path (ref. 7) and is commonly referred to as the first integral of the Euler-Lagrange equations; that is,

$$C = F - \sum_{j=1}^{12} \frac{\partial F}{\partial \dot{\gamma}_j} \dot{\gamma}_j = -(\lambda_1 \dot{u} + \lambda_2 \dot{v} + \lambda_3 \dot{w} + \lambda_4 \dot{x} + \lambda_5 \dot{y} + \lambda_6 \dot{z} - \lambda_7 \beta)$$

$$= \text{const} \qquad (10)$$

where the conditions $F = \dot{c} = \dot{\beta}_{max} = 0$ have been invoked.

In order to solve numerically for an optimum trajectory, the basic two-point boundary value problem must be overcome. This is usually accomplished by using a multivariable Newton-Raphson iteration scheme. In this method, the partial derivatives of the end conditions u, v, w, x, y, and z with respect to the initial values of the λ_i will be needed. These may be obtained along with the solution of equations (1) and (4) by simultaneous integration of the differential equations developed in the appendix. It should be noted here that equations (4) are homogeneous in the λ_i, and, thus, the solution is independent of the initial value of one of the multipliers. The choice, however, sets the numerical scale on the multipliers.

Optimization of Round Trip

As pointed out in reference 6, the Lagrangian multipliers (or, more generally, the expressions for $\partial F/\partial \dot{\gamma}_j$) are the partial derivatives of the function to be extremized with respect to the problem variables. This fact allows the computation of first-order changes in m_f due to changes in the boundary conditions at either end of the trajectory as well as changes in the parameters c and β_{max}. To illustrate how the partial derivative property of the multipliers may be used, consider optimizing the outbound and inbound heliocentric transfers of

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an Earth-Mars round trip (fig. 2(a)). For each transfer, the vehicle is assumed to begin and end with planet kinematic state variables and to be powered during the propulsion phases of the entire journey by a thrustor with constant c and β_{max}. These constraints are the same as those developed in the preceding section for a one-way transfer.

Optimum allocation of heliocentric transfer time. - If the variational solution for an outbound trip of given time has been found by solution of the two-point boundary value problem, and a corresponding inbound transfer has also been found, then, for this pair of reference trajectories (characterized by c, β_{max}, total mission time, wait time, and takeoff date) a neighboring optimal trajectory can be found that differs only from the reference trajectory in outbound travel time t_1. Since equations (1) are satisfied along each outbound trajectory, it follows that

$$ d \int_{t_0}^{t_1} F \, dt = (F \, dt) \Big|_{t_0}^{t_1} + \int_{t_0}^{t_1} \left[\sum_{i=1}^{12} \left(\frac{\partial F}{\partial \dot{\gamma}_i} \delta \dot{\gamma}_i + \frac{\partial F}{\partial \gamma_i} \delta \gamma_i \right) \right] dt = 0 $$

(11a)

The relations

$$ \frac{d}{dt} (\delta \gamma_i) = \delta \dot{\gamma}_i $$

(11b)

are now used in equation (11a) to assist in the integration by parts of the second member on the right to give

$$ 0 = \left(F \, dt + \sum_{i=1}^{12} \frac{\partial F}{\partial \dot{\gamma}_i} \delta \gamma_i \right) \Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} \left\{ \sum_{i=1}^{12} \left[\frac{d}{dt} \left(\frac{\partial F}{\partial \dot{\gamma}_i} \right) - \frac{\partial F}{\partial \gamma_i} \right] \delta \gamma_i \right\} dt $$

(12)

Since equations (3) are also satisfied, the second member of equation (12) vanishes and leaves

$$ 0 = \left(F \, dt + \sum_{i=1}^{12} \frac{\partial F}{\partial \dot{\gamma}_i} \delta \gamma_i \right) \Big|_{t_0}^{t_1} $$

(13a)

In order to allow for arbitrary changes in the end conditions, the equations

$$ d\gamma_i = \dot{\gamma}_i \, dt + \delta \gamma_i \qquad i = 1, 2, \ldots, 12 $$

(13b)

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where the $\dot{\gamma}_i$ are taken from the optimal trajectory, may be introduced into equation (13a) to give

$$0 = \left[\left(F - \sum_{i=1}^{12} \frac{\partial F}{\partial \dot{\gamma}_i} \dot{\gamma}_i\right) dt + \sum_{i=1}^{12} \frac{\partial F}{\partial \dot{\gamma}_i} d\gamma_i\right]_{t_0}^{t_1} \qquad (13c)$$

Since equation (13c) holds along all the trajectories so far considered, it will hold in particular for passage between the end points of the two neighboring extremals considered here. Thus, it follows in this case that

$$-\lambda_7(t_1)dm_1 = (C\ dt + \lambda_1\ du + \lambda_2\ dv + \lambda_3\ dw + \lambda_4\ dx + \lambda_5\ dy + \lambda_6\ dz)_{t_1} \qquad (14)$$

At this point it can be seen that, for $\lambda_7(t_1) = 1$, equation (14) has the form of a total derivative for $m(t_1)$. This fact is an indication that the Lagrangian multipliers are the previously mentioned partial derivatives.

To continue, equation (10) is used with equation (14) to give

$$-\lambda_7(t_1)dm_1 = \left[\lambda_1(du - \dot{u}\ dt) + \lambda_2(dv - \dot{v}\ dt) + \lambda_3(dw - \dot{w}\ dt)\right.$$

$$\left. + \lambda_4(dx - \dot{x}\ dt) + \lambda_5(dy - \dot{y}\ dt) + \lambda_6(dz - \dot{z}\ dt) + \lambda_7\beta\right]_{t_1} \qquad (15)$$

Following reference 7, the differentials dx, du, and so forth belong to the path of the target planet, whereas the other terms are to be taken from the trajectory of the vehicle. Thus,

$$-\lambda_7(t_1)dm_1 = \left[\lambda_1(\dot{u}_p - \dot{u}) + \lambda_2(\dot{v}_p - \dot{v}) + \lambda_3(\dot{w}_p - \dot{w}) + \lambda_4(\dot{x}_p - \dot{x})\right.$$

$$\left. + \lambda_5(\dot{y}_p - \dot{y}) + \lambda_6(\dot{z}_p - \dot{z}) + \lambda_7\beta\right]_{t_1} \qquad (16)$$

where the subscript p is used to designate terms taken from the target planet's path.

Once the two-point boundary value problem has been solved,

$$\left.\begin{array}{l} \dot{x}_p = \dot{x} \\[4pt] \dot{y}_p = \dot{y} \\[4pt] \dot{z}_p = \dot{z} \end{array}\right\} \qquad (17)$$

If the definition of κ is used and it is recognized that the differences $\dot{u}_p - \dot{u}$, $\dot{v}_p - \dot{v}$, and so forth are due only to thrust acceleration, equation (16) becomes

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$$\lambda_7(t_1)dm_1 = (\kappa\beta)_{t_1} dt_1 \qquad (18)$$

Thus, an expression has been derived for the change in target planet arrival mass m_1 due to a change in the outbound travel time. If the mission time and wait time, as given by the reference pair, are considered constant, then a change in t_1 also results in a change in t_2. The resulting change in m_1 also affects m_2 since they are equal for missions with no planetary maneuvers. Thus,

$$t_2 = t_1 + t_w$$

$$t_w = \text{const}$$

$$m_2 = m_1$$

and, therefore,

$$dt_2 = dt_1 \qquad (19a)$$

$$dm_2 = dm_1 \qquad (19b)$$

The inbound trajectory must now be analyzed in a similar manner to determine changes in the Earth return mass m_3 that are caused by changes in m_2 and t_2. From equation (13c) for $t_0 = t_2$ and $t_1 = t_3$,

$$0 = (C\ dt + \lambda_1\ du + \lambda_2\ dv + \lambda_3\ dw + \lambda_4\ dx + \lambda_5\ dy + \lambda_6\ dz + \lambda_7\ dm)\bigg|_{t_2}^{t_3}$$

$$(20)$$

which becomes

$$\lambda_7(t_3)dm_3 = \lambda_7(t_2)dm_2 - (\kappa\beta)_{t_2} dt_2 \qquad (21)$$

where the negative sign occurs because initial conditions, rather than final conditions, are affected. Combining equations (18), (19), and (21) then gives

$$dm_3 = \frac{1}{\lambda_7(t_3)}\left[\frac{\lambda_7(t_2)}{\lambda_7(t_1)}(\kappa\beta)_{t_1} - (\kappa\beta)_{t_2}\right]dt_1 \qquad (22)$$

This equation is then a fundamental equation expressing changes in Earth return mass due to changes in the outbound travel time for a round trip with fixed mission and wait times. On the basis of the sign of the coefficient of dt_1 in equation (22), it is then possible to decide whether or not t_1 should be increased or decreased. Furthermore, since the coefficient will vanish for optimum t_1, it is possible to impose this condition on the initial $\lambda_i(t_2)$ of the inbound transfer. This effectively eliminates one of the six $\lambda_i(t_2)$ from the two-point

boundary value problem and requires that t_3 be used in place of the multiplier to satisfy the six kinematic state variables. Consequently, total mission time becomes a dependent variable, but an optimum round trip is obtained by solving the boundary-value problem once for the outbound trajectory and once for the inbound trajectory. Imposing the condition that the coefficient of dt_1 in equation (22) be zero is a necessary condition but does not guarantee a unique solution, and other means should be used to determine which local optimum has been found.

Optimum c and β_{max}. - In the problem just considered, c and β_{max} were held constant. If they are considered as problem variables, as in the preceding variational solution of the one-way transfer, the transversality condition (eq. (9)) for changes in c and β_{max} only is

$$\left[\lambda_7(t_1) - 1.0\right]dm_1 + \left(\lambda_9 \, dc + \lambda_{10} \, d\beta_{max}\right)\Big|_{t_0}^{t_1} = 0 \qquad (23)$$

where the differentials of the kinematic state variables are zero when initial values of the $\lambda_i(t_0)(i = 1, 2, \ldots, 6)$ are found to satisfy the final values u, v, w, x, y, and z for the target planet. In order that equation (23) be true for arbitrary values of the differentials, it follows that

$$\left.\begin{aligned}\lambda_7(t_1) &= 1.0 \\ \lambda_9(t_0) &= \lambda_9(t_1) \\ \lambda_{10}(t_0) &= \lambda_{10}(t_1)\end{aligned}\right\} \qquad (24)$$

The first expression, $\lambda_7(t_1) = 1.0$, can be satisfied by scaling, as previously explained, since equations (4) are homogeneous in the λ_i. The initial values of λ_9 and λ_{10} can arbitrarily be zero since equations (1) and (4) do not contain these multipliers. The last two expressions can then be satisfied by finding c and β_{max} such that $\lambda_9(t_1) = \lambda_{10}(t_1) = 0$. Since no bounds have been placed on these parameters, however, the final mass will continue to increase with both c and β_{max}, and the conditions (24) will only be satisfied in the limit as both c and β_{max} approach infinite values. This apparent difficulty vanishes once more realistic problems are considered. For example,

$$m_L = m_3 - m_{pp}$$

and

$$m_{pp} = \alpha P_s = \frac{\alpha P_j}{\eta(c)} = \frac{\alpha \beta_{max} c^2}{2\eta(c)} \qquad (25)$$

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where

m_L payload mass

m_{pp} powerplant mass

α factor of proportionality

P_s power supplied to thrustor

P_j jet power

$\eta(c)$ thrustor efficiency (assumed a function of c only)

With these simplifying assumptions and equation (13c), the total effect on m_L due to t_1, c, and β_{max} can be shown to be

$$dm_L = -\left\{\frac{\beta_{max}\alpha c}{2\eta^2}\left(2\eta - c\,\frac{d\eta}{dc}\right) + \frac{1}{\lambda_7(t_3)}\left[\frac{\lambda_7(t_2)}{\lambda_7(t_1)}\lambda_9(t_1) + \lambda_9(t_3)\right]\right\}dc$$

$$-\left\{\frac{\alpha c^2}{2\eta} + \frac{1}{\lambda_7(t_3)}\left[\frac{\lambda_7(t_2)}{\lambda_7(t_1)}\lambda_{10}(t_1) + \lambda_{10}(t_3)\right]\right\}d\beta_{max}$$

$$+ \frac{1}{\lambda_7(t_3)}\left[\frac{\lambda_7(t_2)}{\lambda_7(t_1)}(\kappa\beta)_{t_1} - (\kappa\beta)_{t_2}\right]dt_1 \tag{26}$$

As with equation (24), the signs of the coefficients in equation (26) indicate the directions in which the variables should be changed but not the amount of change. The amounts can be found, however, with an iteration scheme designed to make the three coefficients vanish by appropriate changes in c, β_{max}, and t_1.

<div align="center">Solution for Variable Thrust</div>

Another type of thrustor constraint frequently employed is continuously variable thrust with constant jet power. In order to include this case, equations (1) must be modified by the deletion of f_9 and f_{10} and the replacement of f_8 with

$$\beta c^2 - 2P_j = 0 \tag{27}$$

where P_j is treated as a constant. Actually, to be thoroughly consistent, an equation such as $\dot{P}_j = 0$ should be added. As pointed out in references 1, 3, and 5, though, this thrustor constraint results in trajectories that are indepen-

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dent of P_j and leads to the relation

$$m(t) = \frac{m(0)}{1 + \frac{m(0)}{2P_j} \int_0^t a^2 \, dt} \tag{28}$$

where a, the thrust acceleration of the vehicle, is not a function of P_j. Thus, the best power can be found without the aid of the additional λ_i that would be associated with $\dot{P}_j = 0$.

The introduction of the required modifications into equations (1) results in the following changes in equations (4h) and (4i), respectively:

$$-\frac{c}{m} \left[(\lambda_1 \cos \psi + \lambda_2 \sin \psi)\cos \theta + \lambda_3 \sin \theta \right] + \lambda_8 c^2 + \lambda_7 = 0$$

$$-\frac{\beta}{m} \left[(\lambda_1 \cos \psi + \lambda_2 \sin \psi)\cos \theta + \lambda_3 \sin \theta \right] + \lambda_8 2c\beta = 0$$

which, with the aid of equation (5), become

$$-c\left(\frac{\Lambda}{m} - \lambda_8 c\right) + \lambda_7 = 0$$

$$\beta\left(\frac{\Lambda}{m} - \lambda_8 2c\right) = 0$$

and may be combined to give

$$\lambda_8 = \frac{\Lambda}{2cm} \qquad \beta \neq 0$$

$$\lambda_7 = \frac{c\Lambda}{2m} \tag{29}$$

Furthermore, equations (29), (5), and (4g) can now be combined to give

$$\dot{\lambda}_7 = \frac{c\beta}{m^2} \Lambda = \frac{2\beta}{m} \lambda_7 = -\frac{2\dot{m}\lambda_7}{m}$$

which, after integration, yields

$$\lambda_7 m^2 = \text{const} = K > 0 \tag{30}$$

Finally, equations (27), (29), and (30) allow the thrust acceleration to be written as

$$a = \frac{c\beta}{m} = \frac{2P_j}{cm} = \frac{P_j}{K} \Lambda \tag{31}$$

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As mentioned previously, the acceleration a is independent of P_j and, furthermore, depends only on the chosen boundary conditions. This is most easily demonstrated by substituting equation (29) into equation (10), which gives

$$C = - \left(\lambda_1 \dot{u} + \lambda_2 \dot{v} + \lambda_3 \dot{w} + \lambda_4 \dot{x} + \lambda_5 \dot{y} + \lambda_6 \dot{z} - \frac{\Lambda}{2m} \beta \right) \tag{32}$$

This expression is then solved for the thrust acceleration in the form

$$a = \frac{2 \left[(\lambda_1 V_x + \lambda_2 V_y + \lambda_3 V_z) - (\lambda_4 \dot{x} + \lambda_3 \dot{y} + \lambda_6 \dot{z}) - C \right]}{\Lambda} \tag{33}$$

which can be seen to be a function of time only. All other aspects of the problem are the same with the exception that the Weierstrass test yields only the proper sign for Λ.

When consideration is made of the round trip, t_1 is now the only variable that must be considered, and the variable-thrust feature has no other effect on the preceding development. Then, because of equation (29), the expression for κ can be written as

$$\kappa = \frac{c}{m} \Lambda - \lambda_7 = \frac{c\Lambda}{2m} \tag{34a}$$

Thus, equation (18) becomes

$$dm_1 = \frac{1}{\lambda_7(t_1)} \left(\frac{a\Lambda}{2} \right)_{t_1} dt_1$$

Accordingly, equation (22) becomes

$$dm_3 = \frac{1}{\lambda_7(t_3)} \left[\frac{\lambda_7(t_2)}{\lambda_7(t_1)} \left(\frac{a\Lambda}{2} \right)_{t_1} - \left(\frac{a\Lambda}{2} \right)_{t_2} \right] dt_1 \tag{34b}$$

Equation (31), when introduced into equation (34b) along with equation (30), gives

$$dm_3 = \frac{m_3^2}{2P_j} \left[a^2(t_1) - a^2(t_2) \right] dt_1 \tag{34c}$$

Thus, in the special case of variable thrust, an optimum round trip can be recognized by the fact that the thrust acceleration at the end of outbound transfer is equal to that at the start of the inbound transfer. Here, again, it is possible to impose this condition on the inbound transfer and have the total mission time become a dependent variable.

Inclusion of Planetocentric Maneuvers

Up to this point, the consideration of planetocentric escape and capture maneuvers has been omitted for the sake of clarity. Nevertheless, their inclusion presents no obstacle to the methods so far presented, as will be demonstrated in this section.

The simplest method of including planetocentric maneuvers is to incorporate the duration required for such maneuvers into the waiting phase and consider c and β_{max} as constants. In this special case, equations (19) must be modified so that m_2 is some other function of m_1 that will depend (in form) on the particular maneuver and the method of computation used. Thus, equation (19b) becomes

$$dm_2 = \frac{\partial f(m_1)}{\partial m_1}\, dm_1 \tag{35a}$$

where

$$m_2 = f(m_1) \tag{35b}$$

If this expression had been used in the preceding development, equation (22) would have taken the form

$$dm_3 = \frac{1}{\lambda_7(t_3)} \left[\frac{\lambda_7(t_2)\,\dfrac{\partial f(m_1)}{\partial m_1}}{\lambda_7(t_1)}\,(\kappa\beta)_{t_1} - (\kappa\beta)_{t_2} \right] dt_1 \tag{35c}$$

Like equation (22), this condition can also be imposed on the solution of the inbound trajectory and the optimum mission time found as part of the return-trip solution.

As another example, consider the case of combining, in an optimum fashion, the heliocentric and planetocentric parts (each considered as a separate two-body problem) of a one-way trip using the variable-thrust constraint in all phases. Stated in another way, this is a problem of finding the best values of t_1 and t_2 for given t_0 and t_3 in figure 2(b).

The case of variable thrust has been selected here because a simple approximation that is free of c and β_{max} can be used for escape or capture maneuvers. In particular, it is reported in reference 1 that variational solutions for this type of trajectory compare well with those using constant-thrust acceleration tangentially directed. This greatly simplifies trajectory computations and allows the approximation

$$\bar{a}^2(t)t \simeq \int_0^t a^2\, dt \tag{36a}$$

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for use in equation (28). This approximation has been used in reference 8, where

charts of $\displaystyle\int_0^t a^2\,dt$ are also presented for a number of such maneuvers.

Variations in the Earth escape time will cause the following changes in the heliocentric initial conditions:

$$dm_1 = -\frac{m_1^2}{2P_j}\left[\bar{a}_1^2 + 2\,\Delta t_1\,\frac{d\bar{a}_1}{d(\Delta t_1)}\,\bar{a}_1\right]dt_1 \tag{36b}$$

$$\Delta t_1 = t_1 - t_0$$

which is derived by using equations (28) and (36a). There will also be similar changes in the initial velocity and positions components, but it will be assumed in this simplified analysis that there is no relative motion between the vehicle and the planet. If the methods of the preceding development are followed, the initial changes are transmitted to the end of the heliocentric transfer by

$$\lambda_7(t_2)dm_2 = \lambda_7(t_1)dm_1 - \frac{a(t_1)^2}{2P_j}K\,dt_1 + \frac{a(t_2)^2 K\,dt_2}{2P_j} \tag{37}$$

which also includes the effects due to varying t_2. Equations (37) and (36b) are then combined to give

$$\lambda_7(t_2)dm_2 = -\left\{\frac{\lambda_7(t_1)m_1^2}{2P_j}\left[\bar{a}_1^2 + 2\,\Delta t_1\bar{a}_1\,\frac{d\bar{a}_1}{d(\Delta t_1)}\right] + \frac{a(t_1)^2 K}{2P_j}\right\}dt_1 + \frac{a(t_2)^2 K\,dt_2}{2P_j}$$

$$\tag{38}$$

Since, $\lambda_7(t_1)m_1^2 = K = \lambda_7(t_2)m_2^2$, equation (38) reduces to

$$dm_2 = \frac{m_2^2}{2P_j}\left\{-\left[\bar{a}_1^2 + a(t_1)^2 + 2\,\Delta t_1\bar{a}_1\,\frac{d\bar{a}_1}{d(\Delta t_1)}\right]dt_1 + a(t_2)^2\,dt_2\right\} \tag{39}$$

The target-planet capture spiral undergoes the initial mass variation dm_2 as well as the initial time change dt_2. Thus,

$$dm_3 = \frac{m_3^2}{2P_j}\left[\bar{a}_2^2 + 2\,\Delta t_2\bar{a}_2\,\frac{d\bar{a}_2}{d(\Delta t_2)}\right]dt_2 + \left(\frac{m_3}{m_2}\right)^2 dm_2 \tag{40}$$

where

$$\Delta t_2 = t_3 - t_2$$

which, together with equation (39), leads to

$$dm_3 = \frac{m_3^2}{2P_j}\left\{-\left[\bar{a}_1^2 + a(t_1)^2 + 2\,\Delta t_1 \bar{a}_1 \frac{d\bar{a}_1}{d(\Delta t_1)}\right]dt_1\right.$$

$$\left. + \left[\bar{a}_2^2 + a(t_2)^2 + 2\,\Delta t_2 \bar{a}_2 \frac{d\bar{a}_2}{d(\Delta t_2)}\right]dt_2\right\} \qquad (41)$$

It can now be seen that the optimum is achieved when the coefficients of dt_1 and dt_2 are both zero. In the case of optimum t_1 this occurs when

$$\bar{a}_1^2 + a(t_1)^2 = -2\,\Delta t_1 \bar{a}_1 \frac{d\bar{a}_1}{d(\Delta t_1)} \qquad (42)$$

Typical plots of $\displaystyle\int_0^t a^2\,dt$ or, equivalently, $\bar{a}^2\,\Delta t$ as a function of Δt,

which can be found in references 1 and 8, show that $d\bar{a}/d(\Delta t)$ is always negative.

These conditions suggest that, given a heliocentric transfer (which then gives $a(t_1)$ and $a(t_2)$), the two companion planetocentric transfers, which together with that heliocentric transfer form an optimum, can be identified directly from graphical or numerical data for $\bar{a}$ and $d\bar{a}/d(\Delta t)$ as functions of Δt. This appears to be an improvement over the method suggested in reference 8 for solving this same problem.

RESULTS

In order to demonstrate the usefulness of the criterion expressed by equation (22), let it be shown that the expression vanishes for maximum m_3. For this example, a total mission time of 940 days and a waiting time at Mars of 510 days are chosen along with a starting date of April 9, 1969 (Julian date, 244 0320.5).

The trajectories for this study were computed on a 7090 computer at the Lewis Research Center. All integrations were performed with the Runge-Kutta numerical technique using a step-size control to limit truncation error. Furthermore, the position and velocity components of Earth and Mars, which are needed in this instance as starting and target data, were taken from curve fitted ephemeris data stored on tape. (For further details on this computational system, see ref. 9.) The two-point boundary value problem associated with each inbound and

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outbound trip was overcome by using a multivariable Newton-Raphson scheme, and the required partial derivatives were obtained by integration of the equations given in the appendix along with equations (1) and (4) of the ANALYSIS.

The propulsion system was assumed to be of the intermittent-thrust type operating at a specific impulse of 8000 seconds and at an initial $(t = t_0)$ thrust-weight ratio of 1×10^{-4}.

Figure 3 shows the coefficient of dt_1 in equation (22) and m_3 as a function of the outbound travel time. This figure shows that the computed coefficient behaves exactly as $\partial m_3 / \partial t_1$ (as can be verified by numerical differentiation of the curve for m_3) and thus passes through zero at the maximum value of m_3.

On the basis of the results shown in figure 3, making either equation (22) or (35c) equal zero at the start of the inbound transfer will result in a pair of trajectories that together form an optimum. This has been done for two-dimensional transfers between circular orbits, including the effects of the planetocentric maneuvers. (For details of the one-way, two-dimensional solution used, see ref. 4.) Because of the circular-orbit assumption, both the outbound travel time and angle are free for optimization, and there result two expressions similar to equation (35c), which must be made to vanish at the start of the inbound transfer. Thus, both the inbound travel time and angle are found as part of the inbound boundary value problem, and the mission time and waiting time (which includes the maneuvers about the target planet) are determined as dependent quantities. The details of a typical Earth-Mars solution of this kind are illustrated in figure 4. In this example, the mission is assumed to begin and end in a circular orbit about Earth at 1.10 Earth radii and to maneuver into and out of a circular orbit about Mars at 1.10 Mars radii. All planetocentric maneuvers as well as such terms as $\partial m_5 / \partial m_2$ were calculated with the aid of a semi-empirical approximate solution for escape and capture spirals. The propulsion system used for the entire mission is again assumed to be of the intermittent-thrust type and is characterized, when operating, by a specific impulse of 12,000 seconds and a thrust-weight ratio of 3×10^{-4} at position 1 of figure 4.

When solutions of this type are computed, the independent parameters are the outbound travel time and angle. The inbound transfer is then specified by demanding that the optimality conditions be satisfied at the start. It has been found, however, that there are, at most, two (excluding multiple revolutions about the Sun) inbound transfers that meet the specifications; thus, there result, at most, two distinctly different round trips for each choice of outbound transfer.

CONCLUDING REMARKS

Most important, the method presented allows the straightforward mathematical determination of certain partial derivatives that vanish when a maximum-payload round-trip trajectory has been attained. Once these expressions are known and set equal to zero, they become additional relations between quantities and parameters belonging to both the inbound and outbound transfers of an optimal round trip. Thus, it becomes probable that these added relations can be used to re-

strict the required computations to those transfers that belong together in an
optimum mission. Should it not be possible to so restrict the computations (be-
cause of computational difficulties), the method will at least allow rapid, di-
rect, and more accurate computation of desired partial derivatives that might
otherwise require finite-difference evaluation.

Finally, the proposed method is not restricted to interplanetary-transfer
problems and may be useful for other variational problems associated with flight
mechanics.

Lewis Research Center
    National Aeronautics and Space Administration
        Cleveland, Ohio, November 21, 1962

APPENDIX - DIFFERENTIAL EQUATIONS FOR BOUNDARY VALUE PROBLEM

In most variational solutions, the two-point boundary value problem is usually overcome by use of a multivariable Newton-Raphson scheme. In such methods it will be necessary to have terms such as $\partial x(t_f)/\partial \lambda_i(0)$. A rapid way of obtaining very accurate values for these quantities is integration of a set of differential equations for them. These differential equations can be obtained by direct differentiation of equations (1) and (4) with respect to the parameters $\lambda_i(0)$. First, however, equation (5) is introduced into the system to give

$$
\left.
\begin{aligned}
\dot{u} &= -V_x + \frac{c\beta}{m}\frac{\lambda_1}{\Lambda} & \dot{\lambda}_2 &= -\lambda_5 \\[2mm]
\dot{v} &= -V_y + \frac{c\beta}{m}\frac{\lambda_2}{\Lambda} & \dot{\lambda}_3 &= -\lambda_6 \\[2mm]
\dot{w} &= -V_z + \frac{c\beta}{m}\frac{\lambda_3}{\Lambda} & \dot{\lambda}_4 &= \lambda_1 V_{xx} + \lambda_2 V_{yx} + \lambda_3 V_{zx} \\[2mm]
\dot{x} &= u & \dot{\lambda}_5 &= \lambda_1 V_{xy} + \lambda_2 V_{yy} + \lambda_3 V_{zy} \\[2mm]
\dot{y} &= v & \dot{\lambda}_6 &= \lambda_1 V_{xz} + \lambda_2 V_{yz} + \lambda_3 V_{zz} \\[2mm]
\dot{z} &= w & \dot{\lambda}_7 &= \frac{c\beta}{m^2}\Lambda \\[2mm]
\dot{m} &= -\beta & \dot{\lambda}_9 &= -\frac{\beta}{m}\Lambda \\[2mm]
\ddot{\lambda}_1 &= -\lambda_4 & \dot{\lambda}_{10} &= \beta\lambda_8
\end{aligned}
\right\}
\qquad (A1)
$$

where

$$
V = -\frac{\mu}{\sqrt{x^2 + y^2 + z^2}}
$$

$$
\Lambda = \sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}
$$

$$
\lambda_8 = \frac{\frac{c}{m}\Lambda - \lambda_7}{(\beta_{max} - 2\beta)} = \frac{\kappa}{(\beta_{max} - 2\beta)}
$$

Differentiation of equations (A1) with respect to $\lambda_i(0)$ gives

$$D_i\dot{u} = - (V_{xx}D_ix + V_{yx}D_iy + V_{zx}D_iz)$$

$$+ \frac{c\beta}{m}\left[\left(1 - \frac{\lambda_1^2}{\Lambda^2}\right)\frac{D_i\lambda_1}{\Lambda} - \frac{\lambda_1}{\Lambda^3}(\lambda_2 D_i\lambda_2 + \lambda_3 D_i\lambda_3)\right] - \frac{c\beta}{m^2}\frac{\lambda_1}{\Lambda}D_im + \frac{c}{m}\frac{\lambda_1}{\Lambda}D_i\beta$$

$$D_i\dot{v} = - (V_{xy}D_ix + V_{yy}D_iy + V_{zy}D_iz)$$

$$+ \frac{c\beta}{m}\left[\left(1 - \frac{\lambda_2^2}{\Lambda^2}\right)\frac{D_i\lambda_2}{\Lambda} - \frac{\lambda_2}{\Lambda^3}(\lambda_1 D_i\lambda_1 + \lambda_3 D_i\lambda_3)\right] - \frac{c\beta}{m^2}\frac{\lambda_2}{\Lambda}D_im + \frac{c}{m}\frac{\lambda_2}{\Lambda}D_i\beta$$

$$D_i\dot{w} = - (V_{xz}D_ix + V_{yz}D_iy + V_{zz}D_iz)$$

$$+ \frac{c\beta}{m}\left[\left(1 - \frac{\lambda_3^2}{\Lambda^2}\right)\frac{D_i\lambda_3}{\Lambda} - \frac{\lambda_3}{\Lambda^3}(\lambda_1 D_i\lambda_1 + \lambda_2 D_i\lambda_2)\right] - \frac{c\beta}{m^2}\frac{\lambda_3}{\Lambda}D_im + \frac{c}{m}\frac{\lambda_3}{\Lambda}D_i\beta$$

$$D_i\dot{x} = D_iu \qquad D_i\dot{\lambda}_1 = -D_i\lambda_4$$

$$D_i\dot{y} = D_iv \qquad D_i\dot{\lambda}_2 = -D_i\lambda_5$$

$$D_i\dot{z} = D_iw \qquad D_i\dot{\lambda}_3 = -D_i\lambda_6$$

$$D_i\dot{m} = -D_i\beta$$

$$D_i\dot{\lambda}_4 = V_{xx}D_i\lambda_1 + V_{xy}D_i\lambda_2 + V_{xz}D_i\lambda_3 + (\lambda_1 V_{xxx} + \lambda_2 V_{xxy} + \lambda_3 V_{xxz})D_ix$$

$$+ (\lambda_1 V_{yxx} + \lambda_2 V_{yxy} + \lambda_3 V_{yxz})D_iy + (\lambda_1 V_{zxx} + \lambda_2 V_{zxy} + \lambda_3 V_{zxz})D_iz$$

$$D_i\dot{\lambda}_5 = V_{yx}D_i\lambda_1 + V_{yy}D_i\lambda_2 + V_{yz}D_i\lambda_3 + (\lambda_1 V_{xyx} + \lambda_2 V_{xyy} + \lambda_3 V_{xyz})D_ix$$

$$+ (\lambda_1 V_{yyx} + \lambda_2 V_{yyy} + \lambda_3 V_{yyz})D_iy + (\lambda_1 V_{zyx} + \lambda_2 V_{zyy} + \lambda_3 V_{zyz})D_iz$$

$$D_i\dot{\lambda}_6 = V_{zx}D_i\lambda_1 + V_{zy}D_i\lambda_2 + V_{zz}D_i\lambda_3 + (\lambda_1 V_{xzx} + \lambda_2 V_{xzy} + \lambda_3 V_{xzz})D_ix$$

$$+ (\lambda_1 V_{yzx} + \lambda_2 V_{yzy} + \lambda_3 V_{yzz})D_iy + (\lambda_1 V_{zzx} + \lambda_2 V_{zzy} + \lambda_3 V_{zzz})D_iz$$

$$D_i\dot{\lambda}_7 = \frac{c\beta}{m^2\Lambda}(\lambda_1 D_i\lambda_1 + \lambda_2 D_i\lambda_2 + \lambda_3 D_i\lambda_3) + \frac{c\Lambda}{m^2}D_i\beta - \frac{2c\beta\Lambda}{m^3}D_im$$

$$D_i\dot{\lambda}_9 = - \frac{\beta}{m\Lambda}(\lambda_1 D_i\lambda_1 + \lambda_2 D_i\lambda_2 + \lambda_3 D_i\lambda_3) - \frac{\Lambda}{m}D_i\beta + \frac{\beta\Lambda}{m^2}D_im$$

$$D_i\dot{\lambda}_{10} = \frac{-\beta}{(\beta_{max} - 2\beta)}\left[D_i\lambda_7 - \frac{c}{\Lambda m}(\lambda_1 D_i\lambda_1 + \lambda_2 D_i\lambda_2 + \lambda_3 D_i\lambda_3)\right.$$

$$\left. + c\frac{\Lambda}{m^2}D_im\right] + \frac{\kappa\beta_{max}}{(\beta_{max} - 2\beta)^2}D_i\beta$$

(A2)

where

$$D_i = \frac{\partial}{\partial \lambda_i(0)}$$

$$V_{r_i} = \frac{\mu}{R^3} r_i \qquad r_1 = x, \ r_2 = y, \ r_3 = z$$

$$V_{r_i r_i} = \frac{\mu}{R^3}\left[1 - 3\left(\frac{r_i}{R}\right)^2\right]$$

$$V_{r_i r_j} = -\frac{3\mu}{R^5} r_i r_j \qquad i \neq j$$

$$V_{r_i r_j r_j} = -\frac{3\mu}{R^5} r_i \left[1 - 5\left(\frac{r_j}{R}\right)^2\right] \qquad i \neq j$$

$$V_{r_i r_j r_k} = \frac{15\mu r_i r_j r_k}{R^7} \qquad i \neq j \neq k$$

$$V_{r_i r_i r_i} = -\frac{9\mu r_i}{R^5}\left[1 - \frac{5}{3}\left(\frac{r_i}{R}\right)^2\right]$$

$$R = \sqrt{x^2 + y^2 + z^2}$$

Integration of equations (A2) begins with the boundary conditions

$$D_i \lambda_j(0) = 0 \qquad i \neq j$$

$$D_i \lambda_j(0) = 1 \qquad i = j$$

$$D_i r_j(0) = 0$$

Difficulty is encountered in equations (A2) when attempting to evaluate $D_i\beta$ at points of discontinuity in $\beta(t)$. Considering two neighboring trajectories that differ only in the initial value of one of the λ_i (fig. 5), it can be seen that

$$D_i\beta = \lim_{\Delta\lambda_i(0)\to 0} \frac{\beta\left[\lambda_i(0) + \Delta\lambda_i(0),t\right] - \beta\left[\lambda_i(0),t\right]}{\Delta\lambda_i(0)} \qquad (A3)$$

will be zero at all points except those corresponding to $\kappa(t) = 0$, at which it will be unbounded. This brief, but infinite, pulse will cause a jump discontinuity in all the $D_i \gamma_j$ that contain $D_i \beta$ in their differential equations.

The magnitude of this jump can be evaluated by considering the expression

$$D_i \gamma_j(t_0) = \lim_{\Delta\lambda_i(0) \to 0} \frac{\gamma_j(t_3) - \gamma_j(t_0)}{\Delta\lambda_i(0)} \tag{A4}$$

where the subscript on t refers to the positions indicated in figure 5.

For sufficiently small $\Delta\lambda_i(0)$,

$$\gamma_j(t_3) \simeq \gamma_j(t_0) + \dot{\gamma}_j^*(t_0)(t_1 - t_0) + D_i \dot{\gamma}_j^*(t_1)\Delta\lambda_i(0) + \dot{\gamma}_j(t_2)(t_3 - t_2) \tag{A5}$$

where the asterisk indicates values taken with propulsion on $(\kappa > 0)$; however,

$$\frac{\kappa(t_1)}{\dot{\kappa}(t_0)} \simeq (t_1 - t_0) = (t_2 - t_3)$$

and

$$\kappa(t_1) \simeq -D_i^* \kappa(t_1)\Delta\lambda_i(0) \tag{A6}$$

Equations (A4), (A5), and (A6) then combine to give

$$D_i \gamma_j(t_0) = \lim_{\Delta\lambda_i(0) \to 0} \left\{ D_i^* \gamma_j(t_1) + \frac{D_i^* \kappa(t_1)}{\dot{\kappa}(t_0)} \left[\dot{\gamma}_j(t_2) - \dot{\gamma}_j^*(t_0) \right] \right\} \tag{A7}$$

In the limit as $\Delta\lambda_i(0)$ approaches 0, t_2 approaches t_0, but $\dot{\gamma}_j(t_2)$ does not approach $\dot{\gamma}_j^*(t_0)$ because one of them excludes propulsive effects. This is true unless the $\dot{\gamma}_j(t)$ is not affected by propulsion, as is the case with κ; that is,

$$\dot{\kappa} = \frac{c}{m\Lambda} (\lambda_1\dot{\lambda}_1 + \lambda_2\dot{\lambda}_2 + \lambda_3\dot{\lambda}_3) + \frac{c\beta}{m^2} \Lambda - \dot{\lambda}_7$$

and

$$\dot{\lambda}_7 = \frac{c\beta}{m^2} \Lambda \tag{A8}$$

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Thus,

$$\dot{\kappa} = \frac{c}{m\Lambda} \left(\lambda_1 \dot{\lambda}_1 + \lambda_2 \dot{\lambda}_2 + \lambda_3 \dot{\lambda}_3 \right)$$

which is unaffected by the value of β. Thus, equation (A7) becomes in the limit

$$D_i \Upsilon_j - D_i^* \Upsilon_j = \frac{(\dot{\Upsilon}_j - \dot{\Upsilon}_j^*)}{\dot{\kappa}} D_i^* \kappa \qquad (A9)$$

The term $D_i^* \kappa$ is evaluated by differentiation of

$$\kappa = \frac{c}{m} \Lambda - \lambda_7$$

that is,

$$D_i^* \kappa = \frac{c}{m\Lambda} \left(\lambda_1 D_i^* \lambda_1 + \lambda_2 D_i^* \lambda_2 + \lambda_3 D_i^* \lambda_3 \right) - D_i^* \lambda_7 - \frac{c\Lambda}{m^2} D_i^* m \qquad (A10)$$

Since $D_i^* \beta = 0$, equations (A2) indicate that $D_i^* m = 0$ (or, more generally, a constant), and equation (A10) becomes

$$D_i^* \kappa = \frac{c}{m\Lambda} \left(\lambda_1 D_i^* \lambda_1 + \lambda_2 D_i^* \lambda_2 + \lambda_3 D_i^* \lambda_3 \right) - D_i^* \lambda_7 \qquad (A11)$$

It is interesting to note, however, that $D_i \kappa$ has the same value after the jumps have occurred. This can be seen when equation (A9) for $\Upsilon_j = m$, and λ_7 are substituted into equation (A10). This gives

$$D_i \kappa = \frac{c}{m\Lambda} \left(\lambda_1 D_i \lambda_1 + \lambda_2 D_i \lambda_2 + \lambda_3 D_i \lambda_3 \right) - D_i^* \lambda_7 - \frac{D_i^* \kappa}{\dot{\kappa}} \left[(\dot{\lambda}_7 - \dot{\lambda}_7^*) + \frac{c\Lambda}{m^2} (\dot{m} - \dot{m}^*) \right]$$

$$(A12)$$

Use of equations (A8) and $\dot{m} = -\beta$, though, shows that the last term in equation (A12) vanishes.

Thus, by proceeding as indicated, all jump discontinuities can be evaluated each time κ passes through zero.

In the case of continuously variable thrust, the equations are simpler and free of jump discontinuities. Equation (33) simplifies the components of thrust acceleration so that the first three of equations (A2) become

$$D_i\dot{u} = -\ (V_{xx}D_ix + V_{yx}D_iy + V_{zx}D_iz) + \frac{P_j}{K}\,D_i\lambda_1$$

$$D_i\dot{v} = -\ (V_{xy}D_ix + V_{yy}D_iy + V_{yz}D_iz) + \frac{P_j}{K}\,D_i\lambda_2 \qquad\qquad (A13)$$

$$D_i\dot{w} = -\ (V_{xz}D_ix + V_{yz}D_iy + V_{zz}D_iz) + \frac{P_j}{K}\,D_i\lambda_3$$

Furthermore, the last three of equations (A2) are now unnecessary, and the mass (which will depend on $m(0)$ and P_j) can be integrated from

$$-\dot{m} = \beta = \frac{P_j m^2 \Lambda^2}{2K^2} \qquad\qquad (A14)$$

This latter expression comes from a combination of equations (27), (29), and (30). Since the constant $K = m^2(0)\lambda_7(0)$, it may be chosen at will.

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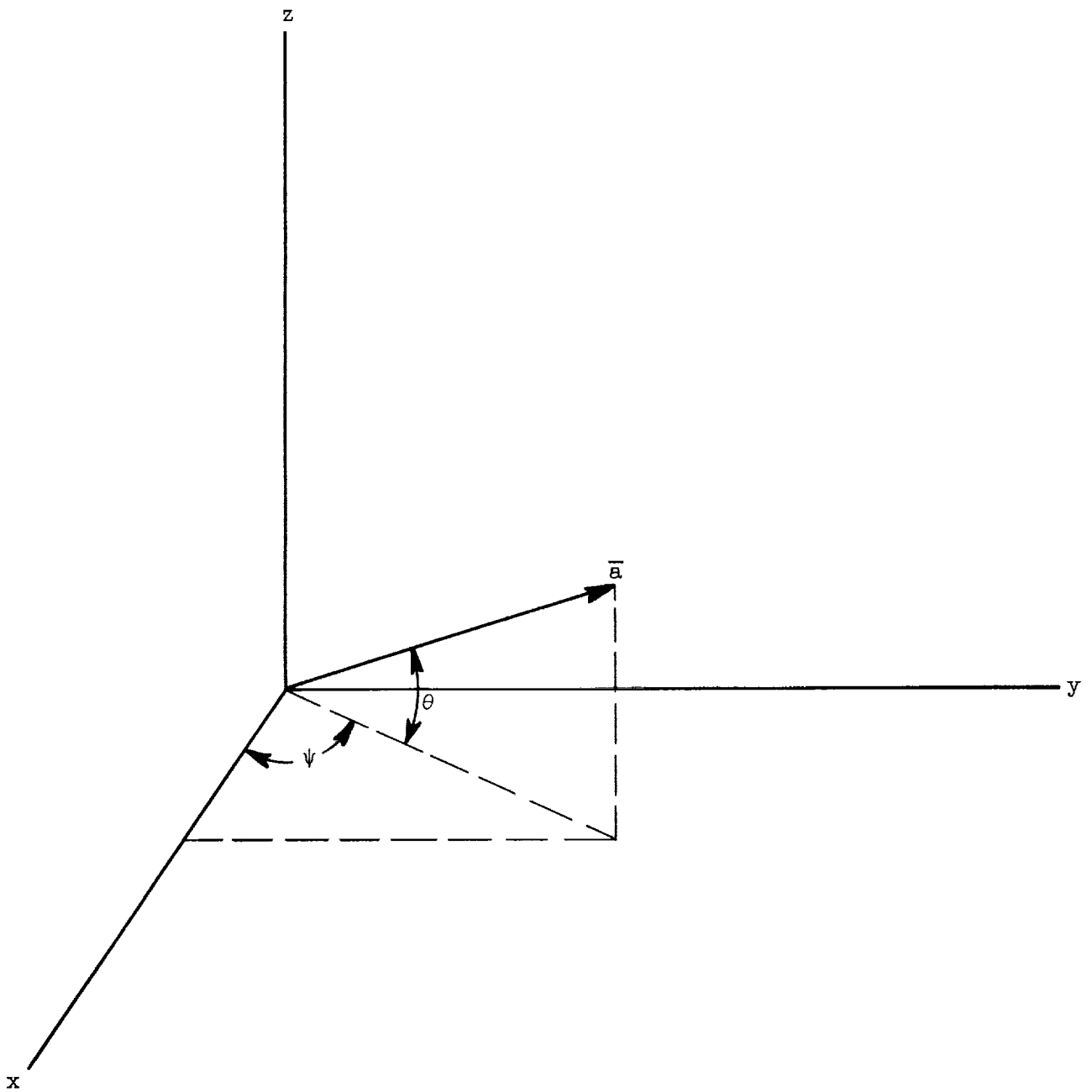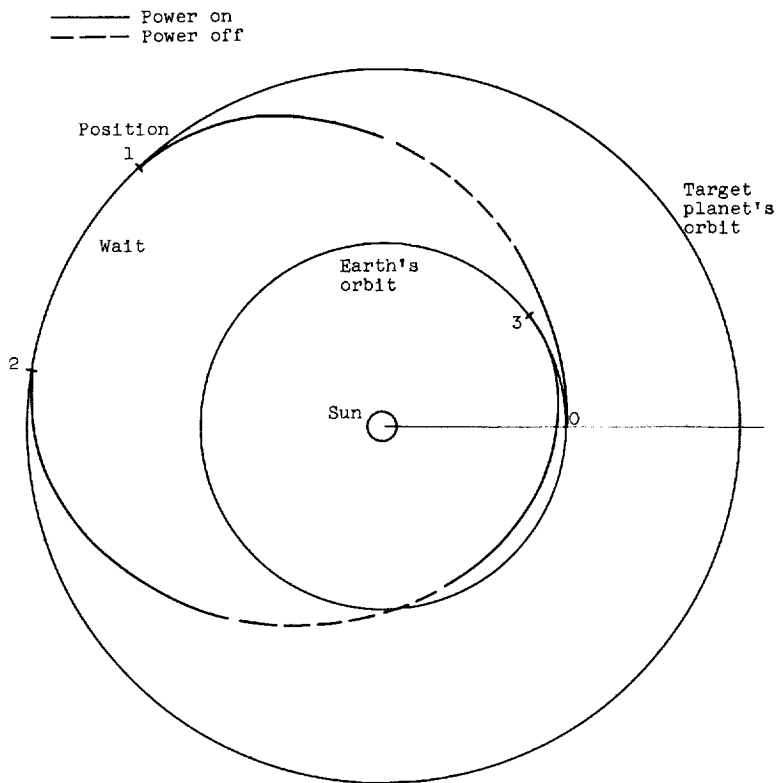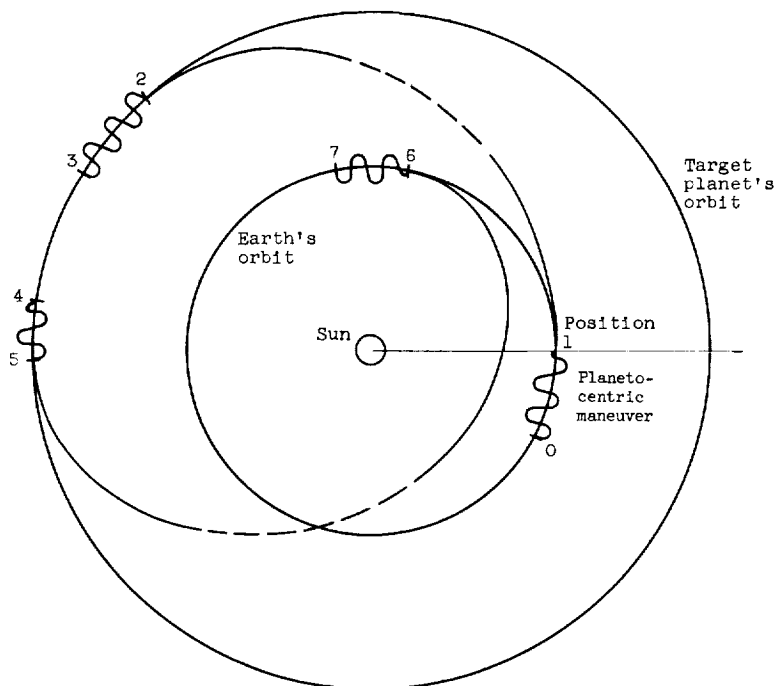
Figure 1. - Orientation of acceleration vector in three dimensions.

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Power on
Power off

Position 1

Wait

Earth's orbit

Target planet's orbit

2

3

Sun

0

(a) Without planetocentric maneuvers.

2

3

7   6

4

5

Earth's orbit

Target planet's orbit

Sun

Position 1

Planeto- centric maneuver

0

(b) Including escape and capture maneuvers.

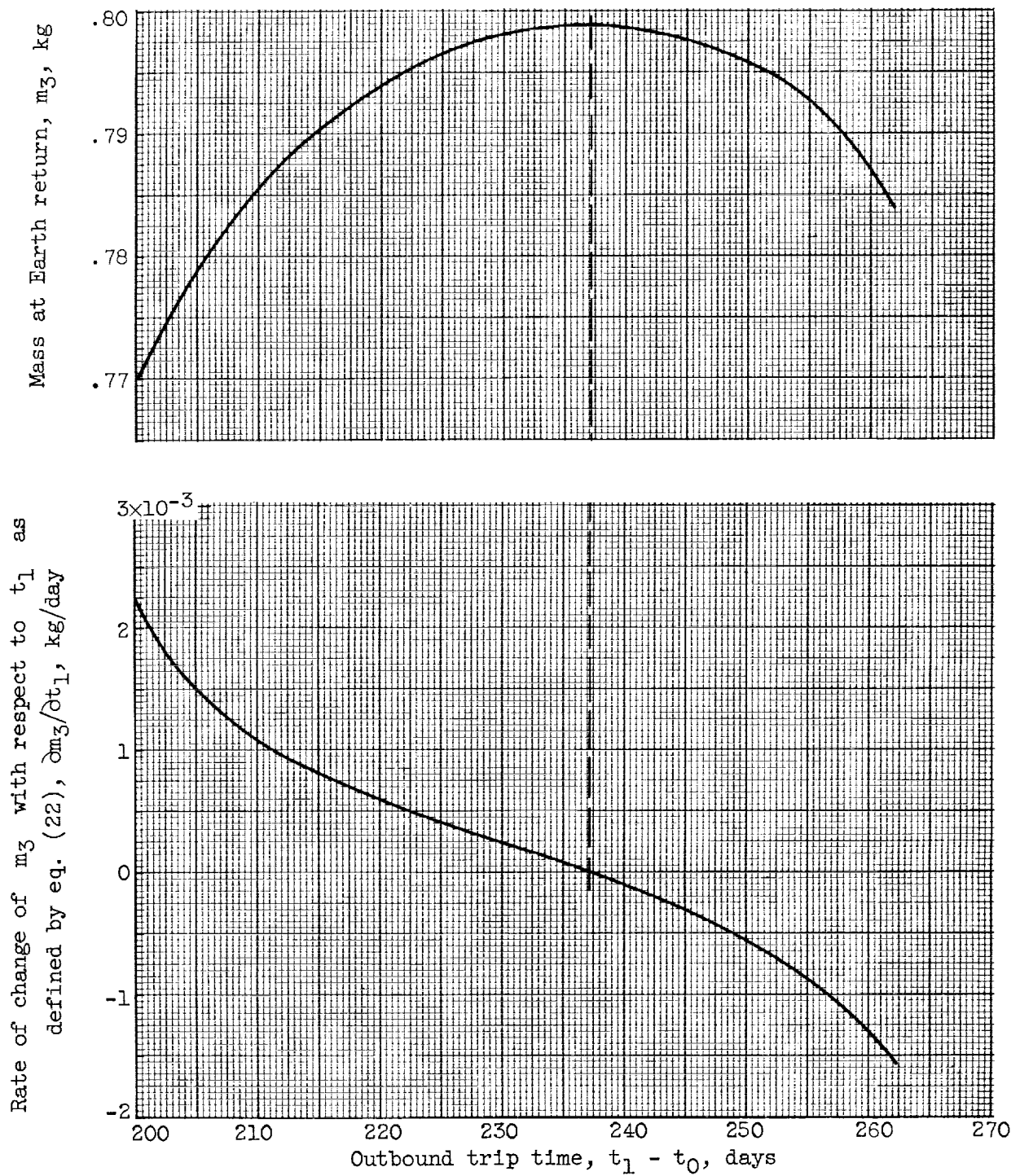Figure 2. - Typical round-trip trajectories.

29

Figure 3. - Effect of outbound travel time on mass at Earth return and rate of change of this mass for Earth-Mars round trip. Mission time, 940 days; wait time, 510 days; thrust-weight ratio, 1×10^{-4}; specific impulse, 8000 seconds; Julian day of takeoff, 244 0320.5; initial mass, 1.0 kilogram.

30

Position	Time, days	Relative mass, $m(t)/m(0)$
0	0	1.00000
1	27.128	.94464
2	139.914	.76066
3	149.034	.74205
4	153.753	.74205
5	162.640	.72392
6	335.713	.51839
7	349.499	.49026

Coast

Propulsion

Position 1

Planetocentric maneuver

Sun

Start

0

7

6

Earth's orbit
(heliocentric radius, 1.495×10^{11} m)

Mars' orbit
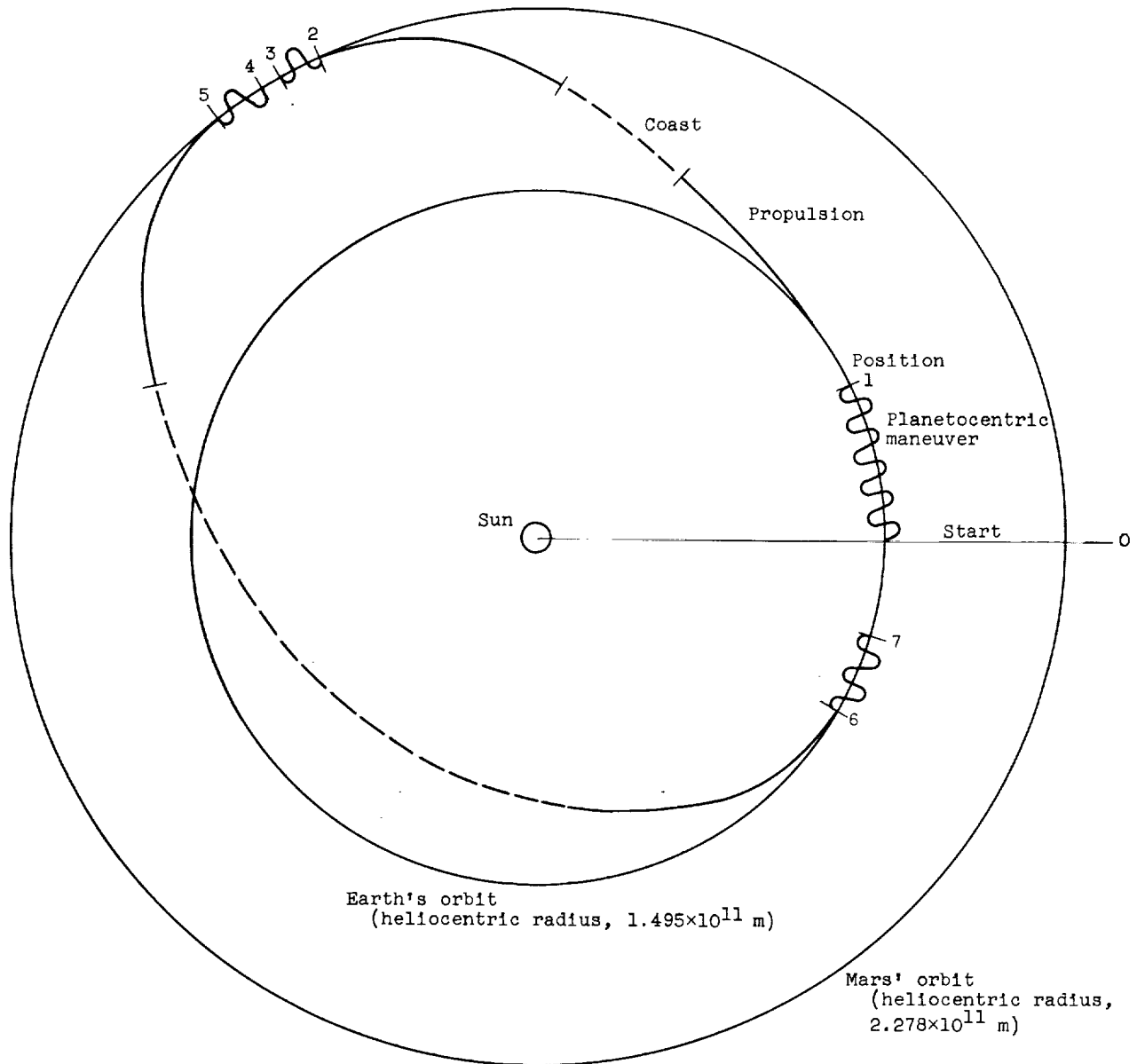(heliocentric radius, 2.278×10^{11} m)

Figure 4. - Typical optimum Earth-Mars round trip in two dimensions between assumed circular orbits with planetocentric maneuvers included. Thrust-weight ratio, 3×10^{-4}; specific impulse, 12,000 seconds; gravitational field strength, 1.3245×10^{20} m^3/sec^2.

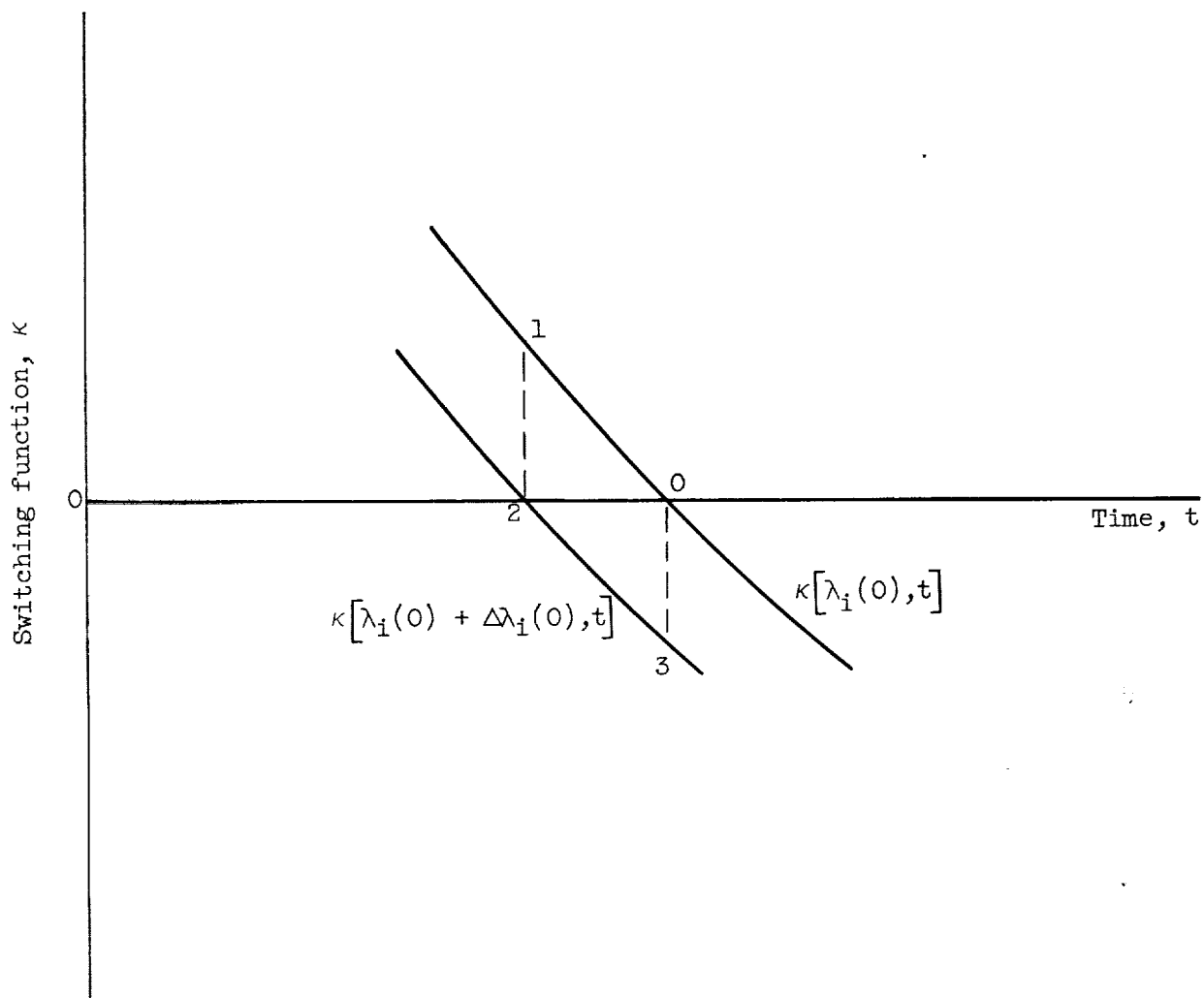Figure 5. - Two neighboring optimal trajectories. Lagrangian multiplier, λ.